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CALCULATION OF INTERFERENCE FOR A POROUS WALL WIND TUNNEL BY THE METHOD OF BLOCK CYCLIC REDUCTION

PROPULSION WIND TUNNEL FACILITY
ARNOLD ENGINEERING DEVELOPMENT CENTER
AIR FORCE SYSTEMS COMMAND
ARNOLD AIR FORCE STATION, TENNESSEE 37389

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The lift interference was calculated for a porous wall wind tunnel by a modified method of block cyclic reduction. This efficient, numerical method has indicated its accuracy by comparison with other available analytical and approximate solutions. A scheme is introduced to search for optimum wall configurations. The lift interference computed for an NACA 64-series finite airfoil in some optimum configurations has

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demonstrated the achievement of minimization of interference. The effect of test section length is also examined.

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PREFACE

The work reported herein was conducted by the Arnold Engineering Development Center (AEDC), Air Force Systems Command (AFSC), under Program Element 65807F. The results of the research presented were obtained by ARO, Inc. (a subsidiary of Sverdrup & Parcel and Associates, Inc.), contract operator of AEDC, AFSC, Arnold Air Force Station, Tennessee. The work was conducted under ARO Project Nos. PF422 and P32A-29A. The authors of this report were C. F. Lo and H. N. Glassman, ARO, Inc. The manuscript (ARO Control No. ARO-PWT-TR-75-63) was submitted for publication on May 21, 1975.

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1.0 INTRODUCTION

The minimization of tunnel wall interferences has become one of the major tasks after the introduction of ventilated transonic tunnels. A variable, but uniformly distributed, porosity wall was designed to reduce interferences at various Mach numbers, e.g., the Aerodynamic Wind Tunnel (4T) at AEDC. The recent requirement for an increase in the size of the testing model to achieve higher Reynolds number creates severe interference which prohibits obtaining useful data. In addition, the axial gradients of interference may cause interference on pitching moment for a long model. By introducing an axially distributed porosity in the walls of a slotted tunnel, the elimination of pitching moment and lift interferences was achieved in the experimental development of walls for V/STOL testing (Ref. 1). It is necessary to search for a theoretical optimum porosity distribution for the minimization of interference as the guideline for an experimental program.

The first theoretical approach to the problem has been carried out in Ref. 2 to reduce the interference in a two-dimensional perforated tunnel by a gaussian type distribution of porosity with an approximate method. Specifically, a system of integral equations was derived using Fourier transform and convolution theorems and then solved by the collocation method with a series form representing the unknown functions. The selection of a gaussian distribution is strictly based on the merits of mathematical simplicity. The reduction of interference is achieved (Ref. 2) by using a simple singularity to represent the test model. This has been extended to a finite chord airfoil to permit comparisons directly with experimental data (Ref. 3). However, the approximate method is limited to certain porosity distributions. The complete elimination of the magnitude and axial gradient of interference requires a nongaussian porosity distribution. To provide such a solution, a numerical method for computing the interference has been developed to search for an optimum configuration in the present study. The application of a modified method of Block Cyclic Reduction (Ref. 4) to the lift interference computation is presented. The scheme to search for an optimum configuration is discussed and extended to a finite airfoil. The lift interference is calculated for an NACA 64-series airfoil in an optimum configuration to demonstrate the achievement of minimization of interference. The effect of test section length is briefly examined.

2.0 GENERAL ANALYSIS

The lift interference in a two-dimensional porous transonic tunnel is formulated for tunnel walls with varying porosity distributions. The optimum porosity distribution may then be obtained by judicious selection for a given application.

2.1 FORMULATION OF MATHEMATICAL PROBLEM

The field equation of an inviscid, irrotational fluid for subsonic flow in terms of the perturbation velocity potential Φ in X-Y coordinates (Fig. 1) is

$$\beta^2 \frac{\partial^2 \Phi}{\partial X^2} + \frac{\partial^2 \Phi}{\partial Y^2} = 0 \tag{1}$$

For the boundary condition of the tunnel, the average mass flow is assumed proportional to the pressure drop across the porous wall as

$$R(x) \frac{\partial \Phi}{\partial X} + \frac{\partial \Phi}{\partial Y} = 0 \qquad \text{at } Y = \pm h \tag{2}$$

where R(x) is the empirical constant, or porosity parameter, of the porous wall and is a function of streamwise location.

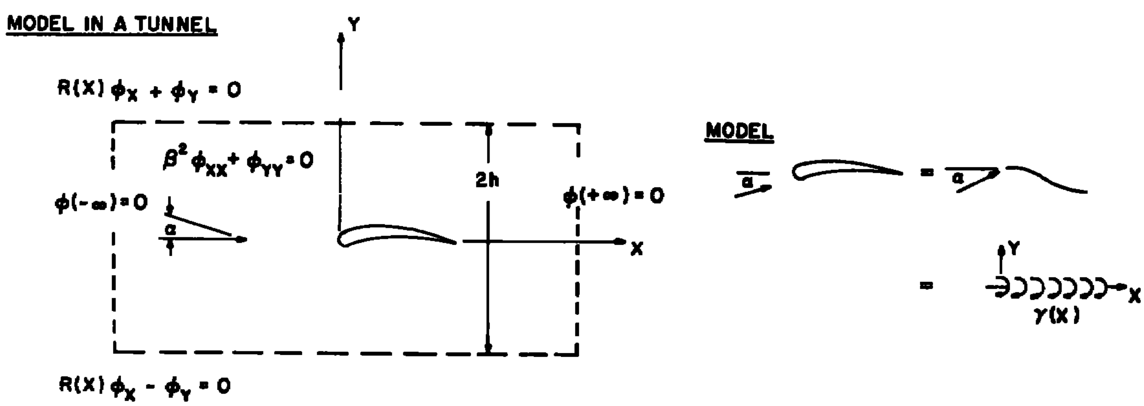

Figure 1. Boundary value problem for tunnel lift interference.

Within the assumptions of linerarized theory, the perturbation velocity potential may be divided into two parts as

$$\Phi = \phi + \phi_m \tag{3}$$

where ϕ is the interference potential caused by the presence of tunnel walls and ϕ_m is the disturbance potential induced by a model. The linearity of the field equation and boundary conditions in the normalized coordinates $x = X/\beta h$, $y = Y/h$ gives

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \tag{4}$$

and

$$\frac{R(x)}{\beta} \frac{\partial \phi}{\partial x} \pm \frac{\partial \phi}{\partial y} = - \left(\frac{R(x)}{\beta} \frac{\partial \phi_m}{\partial x} \pm \frac{\partial \phi_m}{\partial y} \right), \quad y = \pm 1 \tag{5}$$

with the upstream and downstream conditions described as

$$\phi(\pm\infty) = 0 \tag{6}$$

The formulation is completed with the set of Eqs. (4), (5), and (6). The finite difference method will be used to solve this system. An efficient numerical scheme is provided by the modification of Block Cyclic Reduction to yield a solution of the finite difference equations.

2.2 FINITE DIFFERENCE EQUATIONS

To develop the finite difference equations, it is assumed that the interference potential ϕ effectively becomes zero at a large finite distance from the model location. This distance will be denoted x^*.

Consider the rectangular region

$$\bar{R} = \left\{ \begin{matrix} -x^* \le x \le x^* \\ -1 \le y \le 1 \end{matrix} \right\}$$

Let N be any positive integer and let k be any nonnegative integer.

Define $M = 2^k$. Let the region $\bar{R}$ be overlaid with a rectangular net with spacings

$$\delta x_i = x_{i+1} - x_i$$

$$i = 0, \ldots, N-1$$

where the mesh points in the x direction may be distributed as desired. It will be required that $x_o = -x^*$ and $x_N = x^*$.

In the y direction, $\delta y = \frac{1}{M}$ and $y_j = j\delta y \quad j = 0, \pm 1, \ldots, \pm M.$

For notational convenience, column vectors such as

$$\underset{\sim}{u} = \begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_N \end{bmatrix}$$ will be denoted as

$$\underset{\sim}{u} = \text{col}\ (u_1, u_2, \ldots, u_N)$$

and any NxN tridiagonal matrix K of the form

$$K = \begin{bmatrix} b_1 & c_1 & & \\ a_2 & b_2 & c_2 & \\ & & \ddots & \\ & & a_N & b_N \end{bmatrix}$$

will be denoted by $K = (a_i,\ b_i,\ c_i)_{NxN}$.

Let the value of the solution of the finite difference equations at the point (x_i, y_j) be denoted as $\phi_{i,j}$ and let

$$\underset{\sim}{\phi}_{\ell} = \mathrm{col}\left(\phi_{1,\ell},\ \phi_{2,\ell},\ \ldots\ldots,\ \phi_{N-1,\ell}\right) \tag{7}$$

$$\ell = 0,\ \pm 1,\ \ldots.,\ \pm M.$$

Using the centered second difference approximation for $\frac{\partial^2\phi}{\partial x^2}$ and $\frac{\partial^2\phi}{\partial y^2}$ as given in Ref. 5 for variable steps, the finite difference approximation to Eq. (4) on $\bar{R}$ becomes

$$\left(\frac{\phi_{i+1,j}}{\delta x_i(\delta x_i + \delta x_{i-1})} - \frac{\phi_{i,j}}{\delta x_{i-1}\delta x_i} + \frac{\phi_{i-1,j}}{\delta x_{i-1}(\delta x_i + \delta x_{i-1})}\right)$$

$$+ \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{2\delta y^2} = 0 \tag{8}$$

$$i = 1, \ldots.., N-1$$

$$j = 0, \pm 1, \ldots., \pm(M-1)$$

Let $a_i = 2\delta y^2 / \left[\delta x_i \left(\delta x_i + \delta x_{i-1}\right)\right]$

$$b_i = -2\left[1 + \delta y^2 / \left(\delta x_i \delta x_{i-1}\right)\right]$$

$$c_i = 2\delta y^2 / \left[\delta x_{i-1}\left(\delta x_i + \delta x_{i-1}\right)\right]$$

Since it is required that $\phi(-x^*, y) = \phi(x^*, y) = 0$, there results $\phi_{N,j} = \phi_{o,j} = 0$, $j = 0, \pm 1, \ldots, \pm M$. Then Eq. (8) may be written

$$\underset{\sim}{\phi}_{j+1} + A\underset{\sim}{\phi}_j + \underset{\sim}{\phi}_{j-1} = 0$$

where A is the matrix

$$A = \left(C_i,\ b_i,\ a_i\right)_{N-1 \times N-1} \tag{9}$$

The procedure along the boundaries $y = \pm 1$ is as follows:

$$\text{Let } B_i^{\pm} = -\left(\frac{\partial \phi_m(x_i, \pm 1)}{\partial x} \pm T_i \frac{\partial \phi_m(x_i, \pm 1)}{\partial y}\right)$$

where $T_i = \beta / R(x_i)\quad i = 1, 2, \ldots, N-1$

On $y = +1$, the difference approximations given in Ref. 5 are again used to approximate Eq. (5) resulting in

$$\begin{aligned} &p_i \phi_{i+1,M} + q_i\ \phi_{i,M} + r_i \phi_{i-1,M} \\ &\qquad + T_i\ \frac{\phi_{i,M+1} - \phi_{i,M-1}}{2\delta y} = B_i^{+} \end{aligned} \tag{10}$$

where $p_i = \delta x_{i-1} / \left[\delta x_i\ \left(\delta x_i + \delta x_{i-1}\right)\right]$

$q_i = (\delta x_i - \delta x_{i-1}) / (\delta x_i \delta x_{i-1})$

and $r_i = -\ \delta x_i / \left[\delta x_{i-1}\ \left(\delta x_i + \delta x_{i-1}\right)\right]$.

Eq. (4) is also required to hold for $y = +1$ which gives

$$\phi_{i,M+1} + a_i\ \phi_{i+1,M} + b_i\ \phi_{i,M} + c_i\ \phi_{i-1,M} + \phi_{i,M-1} = 0$$

Eliminating $\phi_{i,\,M+1}$ from these two equations yields

$$T\ \ \underset{\sim}{\phi}_{M-1} = E\phi_M + \underset{\sim}{f}^{+} \tag{11}$$

where E is the matrix

$$E = \left[\frac{1}{2}\left(r_i^* - T_i c_i\right),\ \frac{1}{2}\left(q_i^* - T_i b_i\right),\ \frac{1}{2}\left(p_i^* - T_i a_i\right)\right]_{N-1 \times N-1}$$

with $q_i^* = 2\delta y\ q_i$, $p_i^* = 2\delta y\ p_i$, $r_i^* = 2\delta y\ r_i$, $f_i^+ = -\delta y\ B_i^+$

and where

$$T = \left(0,\ T_i,\ 0\right)_{N-1 \times N-1}.$$

In a similar manner it can be shown that on the boundary y = -1,

$$T\underset{\sim}{\phi}_{1-M} = E\underset{\sim}{\phi}_{-M} + f^- \quad . \tag{12}$$

The set of finite difference equations (Eqs. (8), (11), and (12)) is readily solved for the determination of lift interference once the lift potential is established.

3.0 LIFT INTERFERENCE

The lift interference factor is defined by

$$\delta = \frac{C}{SC_L}\ \frac{1}{U}\ \frac{\partial \phi}{\partial y}$$

In particular, the factor along the centerline, y = 0, can be obtained by

$$\delta = \frac{C}{SC_L}\ \frac{1}{U}\ \frac{\underset{\sim}{\phi}_1 - \underset{\sim}{\phi}_{-1}}{2\delta y} \tag{13}$$

where $\underset{\sim}{\phi}_i - \underset{\sim}{\phi}_{-1}$ can be computed by solving an N-1 system of equations using the Modified Method of Block Cyclic Reduction which is described in Appendixes A and B.

3.1 SMALL CHORD AIRFOIL

In the first step, a simple vortex is chosen to represent the lift model as

$$\phi_m = \frac{-\Gamma}{2\pi} \tan^{-1} \frac{y}{x} \tag{14}$$

A solution for the case of a wall with a uniform porosity distribution has been obtained to check with the known analytical solution case and is shown in Fig. 2. The second case, computed for an inverse gaussion distribution of R/β, is compared with results obtained by the approximate method (Ref. 2) in Fig. 3. The agreement between the results using the proposed technique and previous solutions for the above cases indicates that the accuracy of the present numerical solution is satisfactory.

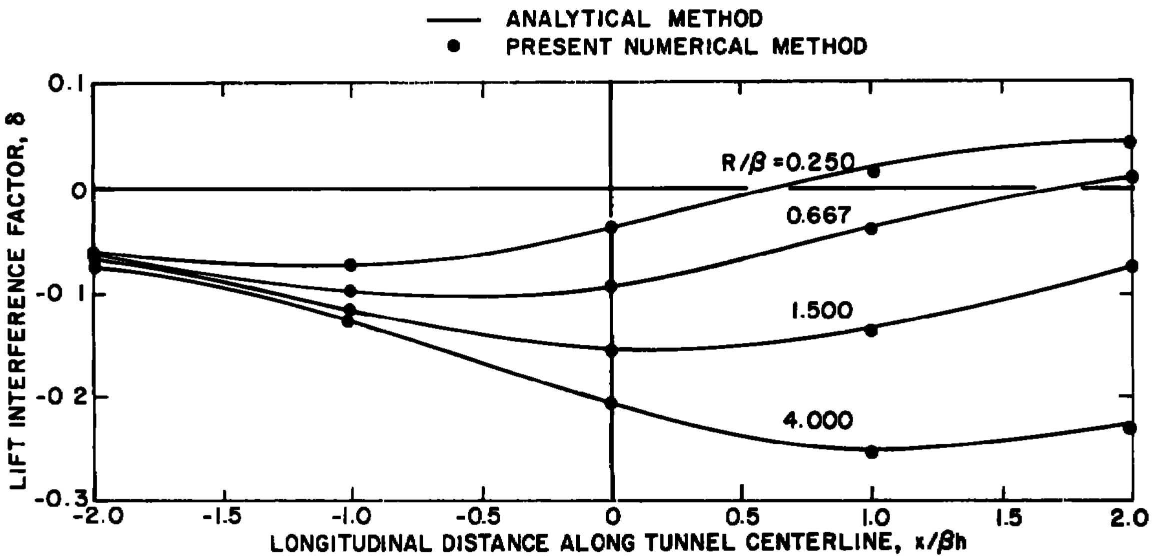

Figure 2. Comparison of block cyclic reduction and analytic solutions for walls with uniform porosity distribution.

3.2 OPTIMUM POROSITY DISTRIBUTION

The ideal porosity distribution for a tunnel wall is defined as that which induces no lift interference anywhere in the test section. In the mathematical sense, the upwash interference, $\partial \phi / \partial y$, vanishes everywhere; or the interference potential is a trivial solution of the system

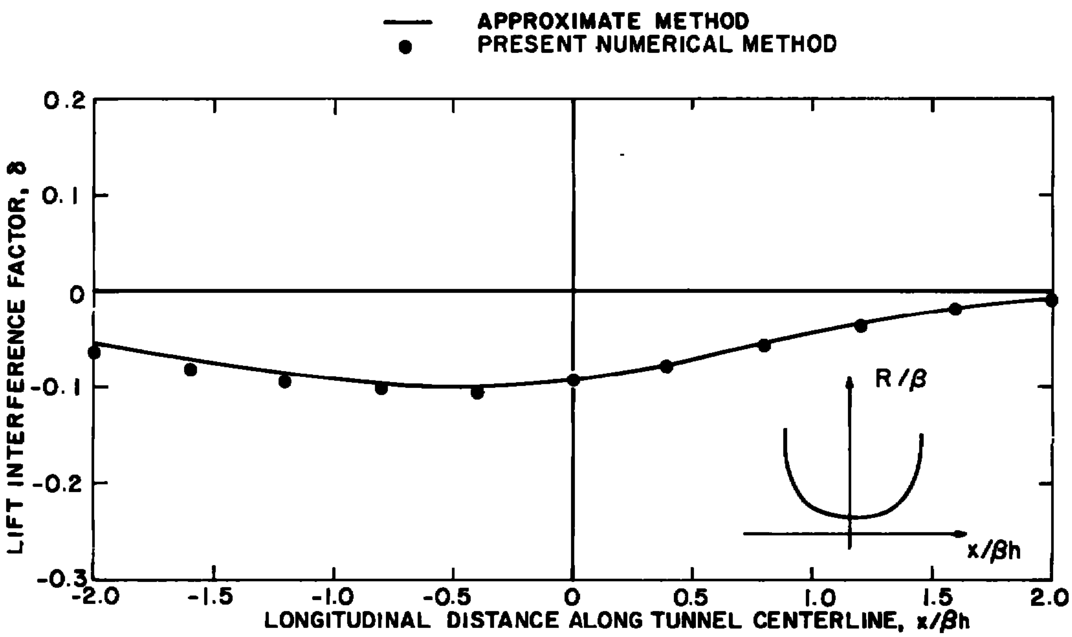

Figure 3. Comparison of block cyclic reduction and approximate solutions for walls with inverse gaussian porosity distribution.

of Eqs. (4), (5), and (6). This solution can be obtained by observation as the right-hand side of Eq. (5) becomes zero and substituting Eq. (4) then

$$R(x)/\beta = \left(\mp \frac{\partial \phi_m}{\partial y} \Big/ \frac{\partial \phi_m}{\partial x}\right)_{y\,=\,\pm 1} \tag{15}$$

$$= x$$

However, the porosity parameter for the perforated wall R/β can only have a positive value because the mass flow is always from the high-pressure to the low-pressure side. Thus, a distribution of $R(x)/\beta$ is selected and shown in Fig. 4 and Table 1 denoted by Configuration C as

$$R(x)/\beta = \begin{cases} x/\beta h & x \geqslant 0 \\ 0 & x \leqslant 0 \end{cases} \tag{16}$$

to evaluate the interference. The lift interference factor for Configuration C has been calculated and is shown in Fig. 5. The interference factors for three additional Configurations D, E, and F (Fig. 4 and Table 1)

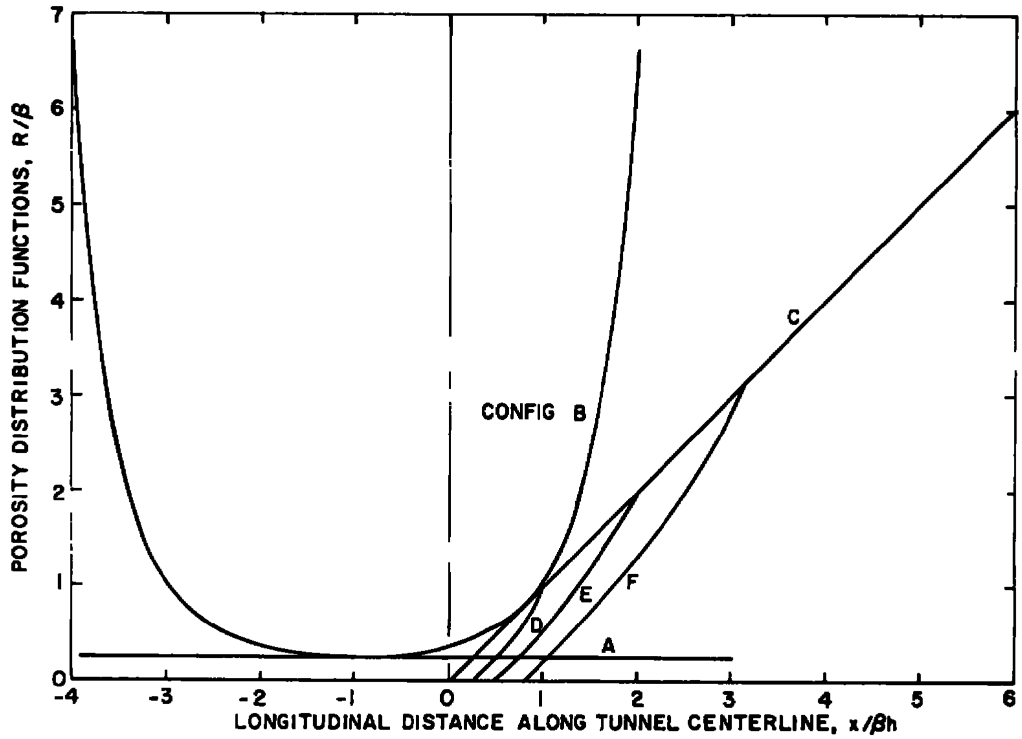

Figure 4. Wall configuration with various porosity distributions.

Table 1. Wall Porosity Distribution, R/β for Various Configurations

X \ CONFIG	C	D	E	F
-17.00	0.000	0.000	0.000	0.000
:	↓	↓	↓	↓
0 00				
0 25	0 250			
0 50	0 500	0.250		
0.75	0 750	0.500	0.250	
1.00	1 000	1.000	0.500	0 200
1.25	1.250	1.250	0.850	0.450
1 50	1.500	1.500	1.200	0.700
1.75	1 750	1.750	1.550	0.975
2.00	2.000	2.000	2.000	1.300
2 25	2.250	2.250	2.250	1.600
2.50	2.500	2.500	2.500	1.950
2.75	2.750	2.750	2.750	2.375
3.00	3.000	3.000	3.000	2.850
3.25	3.250	3.250	3.250	3.250
4.00	4.000	4.000	4.000	4.000
5.00	5.000	5.000	5.000	5.000
17.00	17.000	17.000	17.000	17.000

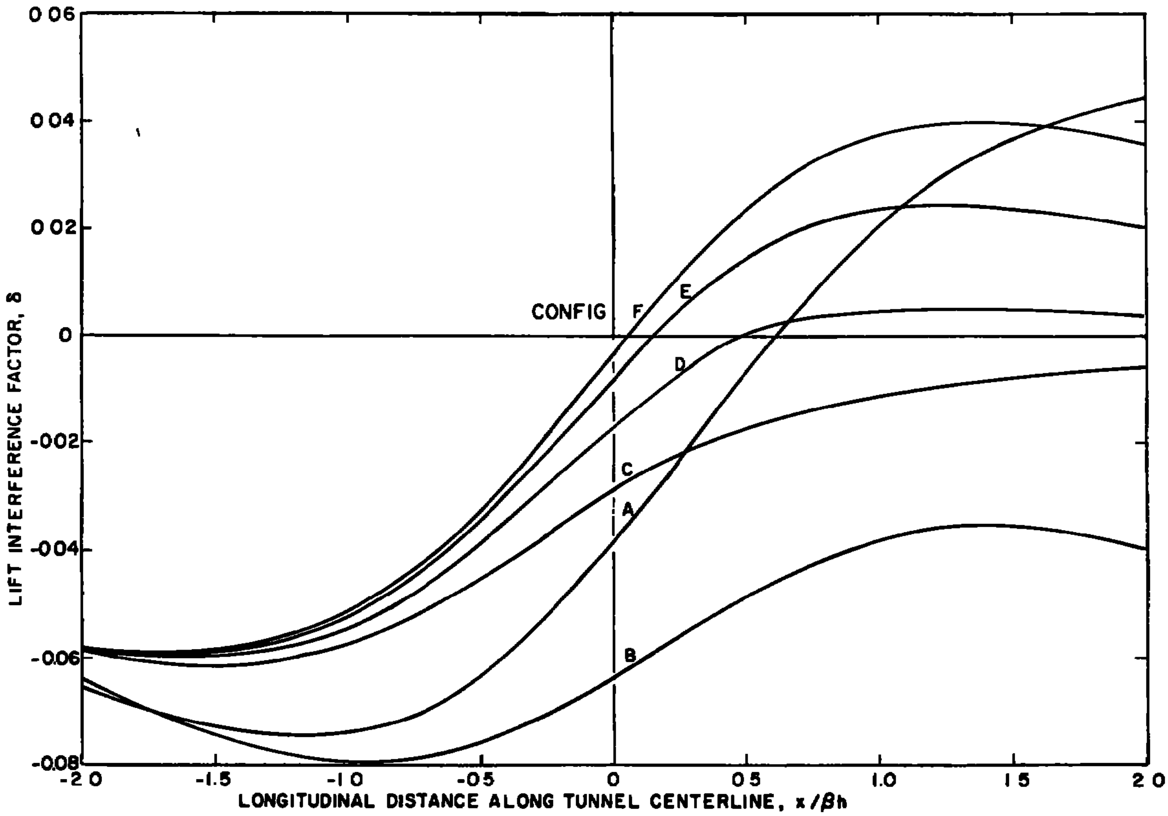

Figure 5. Lift interference on a small chord airfoil in tunnels with various wall configurations.

with a slight variation from Configuration C have been calculated and are presented in Fig. 5. Also plotted in Fig. 5 are results for walls with uniform (Configuration A) and inverse gaussian (Configuration B) porosity distributions. It seems that Configurations C and D give, overall, less interference.

3.3 FINITE CHORD AIRFOIL CASE

For a finite chord airfoil with camber and incidence, a discrete distribution of vortices can be used as

$$\phi_m = \frac{-1}{2\pi} \sum \alpha_j(\zeta_j)\ \Delta\zeta \cdot \tan^{-1} \frac{y}{x - \zeta_j} \tag{17}$$

The results for the NACA 64-series airfoil with a chord $C = 0.5/\beta h$ are presented in Fig. 6 and indicate that Configurations D and E exhibit the most satisfactory distribution of porosity to obtain the minimum interference factor.

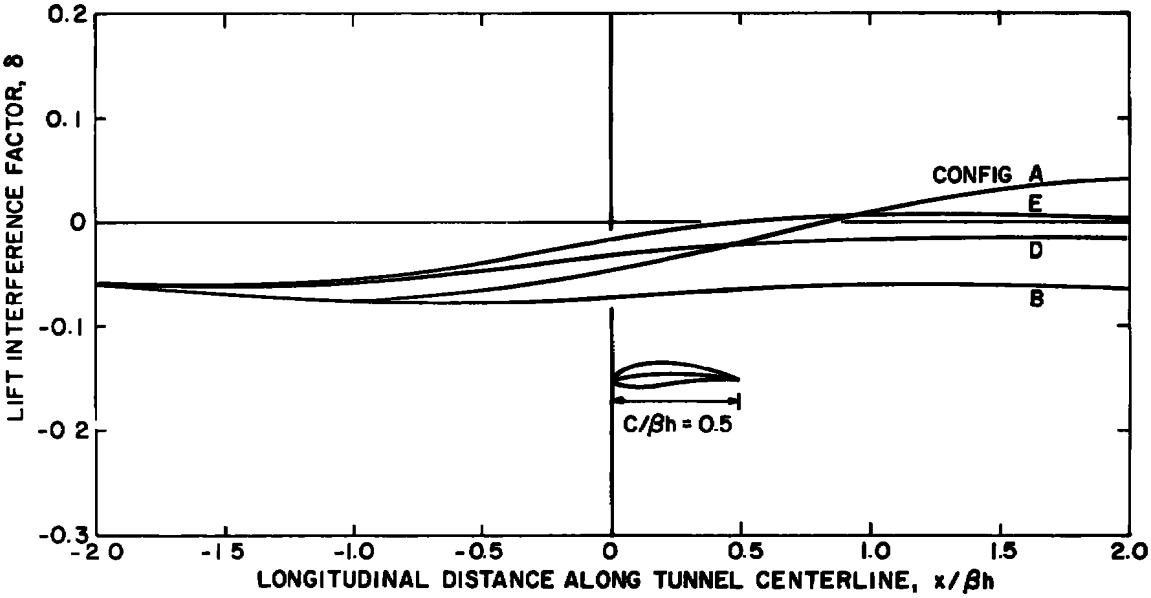

Figure 6. Lift interference on a finite chord airfoil in tunnels with various wall configurations.

3.4 EFFECT OF TEST SECTION LENGTH

Most analytical approaches in wind tunnel theory have assumed the length of test section to be infinite for mathematical simplicity. The effect of test section length on the lift interference is of interest since the actual tunnel test section length is usually about two to three times the test section height. The versatility of the present approach can be applied to examine the effect of test section length. For the uniform porosity distribution case, the comparison of lift interference of a finite test section as $-2 \leqslant x/\beta h \leqslant 3$ (upstream and downstream regions using solid walls) with the infinite test section is shown in Fig. 7 and indicates the effect on the interference in the region $x/\beta h > 2$. It can be seen that the assumption of an infinite length test section for the calculation of interference in the neighborhood of the model appears reasonable.

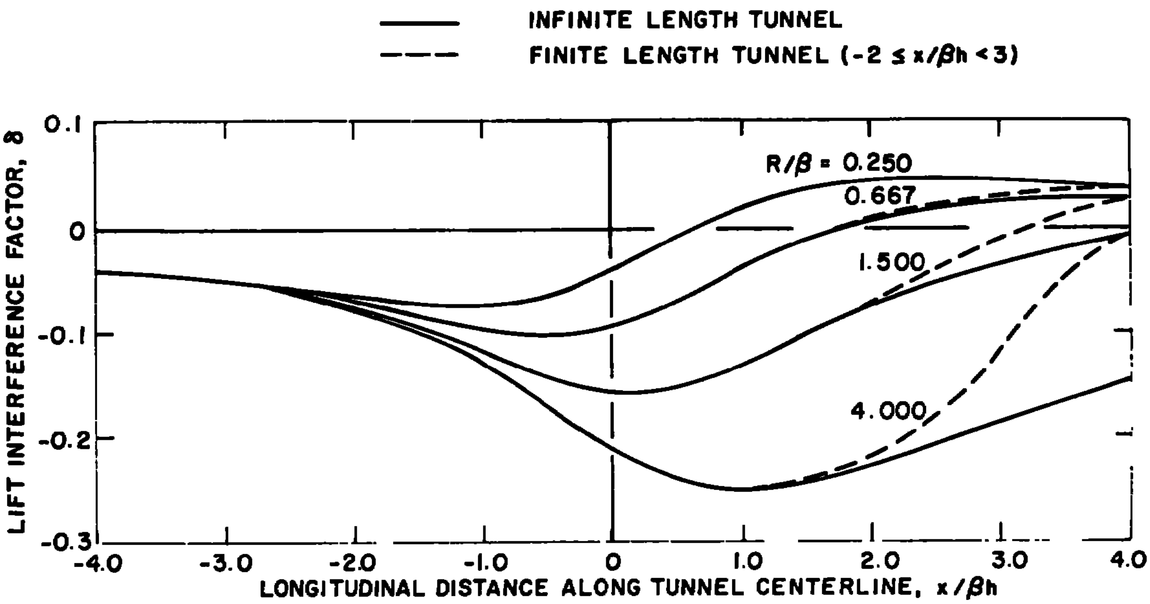

Figure 7. Effect of test section length on lift interference.

4.0 CONCLUDING REMARKS

An efficient numerical scheme has been developed by a modification to the Block Cyclic Reduction Method for computing lift interference in a wind tunnel with an arbitrary distribution of wall porosity. A comparison with other available analytical and approximate solutions has demonstrated the accuracy of the present numerical method. The optimum porosity distribution to minimize interference is obtained by the variation of the ideal mathematical configuration which produces exact interference-free condition. The minimization of interference is also presented for a finite chord airfoil in the optimum wall configurations.

The optimum wall porosity configurations have been calculated for both a simplified ideal airfoil and a finite chord airfoil. The effect of test section length has been also studied.

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APPENDIX A
METHOD OF BLOCK CYCLIC REDUCTION

Consider the problem of solving the finite difference analog to Laplace's equation with the boundary conditions

$$
\begin{aligned}
\phi(-x^*,y) &= \phi(x^*, y) = 0 \qquad -1 \le y \le 1 \\
\phi(x,\pm 1) &= g^{\pm}(x) \qquad\qquad -x^* \le x \le x^*
\end{aligned}
\tag{A-1}
$$

and where $g^{\pm}(x)$ are given functions with

$$g^{\pm}(x^*) = g^{\pm}(-x^*) = 0 .$$

It is well known that replacing Laplace's equation on the region R by a centered second difference approximation and imposing the boundary conditions given in Eq. (A-1) yields the problem

$$D\ \underset{\sim}{\phi} = \underset{\sim}{y} \tag{A-2}$$

where D is the $(2M-1)(N-1) \times (2M-1)(N-1)$ real symmetric matrix which has the block tridiagonal form

$$D = (I,\ A,\ I)_{2M-1 \times 2M-1}$$

and A is the matrix defined in Eq. (9).

The vector $\underset{\sim}{\phi}$ will be given in partitioned form as

$$\underset{\sim}{\phi} = \mathrm{col}\left(\underset{\sim}{\phi}_{M-1},\ \underset{\sim}{\phi}_{M-2},\ \ldots,\ \underset{\sim}{\phi}_{1-M}\right) .$$

Likewise, the vector y is given by

$$\underset{\sim}{y} = \mathrm{col}\left(-\underset{\sim}{\phi}_{M},\ \underset{\sim}{0},\ \ldots.,\ \underset{\sim}{0},\ -\underset{\sim}{\phi}_{-M}\right) .$$

In their description of Block Cyclic Reduction, Buzbee, Golub, and Nielson (Ref. 6) first write Eq. (A-2) as

$$A\underset{\sim}{\phi}_{M-1} + \underset{\sim}{\phi}_{M-2} = \underset{\sim}{Y}_{M-1} \tag{A-3a}$$

$$\underset{\sim}{\phi}_{j+1} + A\underset{\sim}{\phi}_{j} + \underset{\sim}{\phi}_{j-1} = 0 \quad j=0,\pm 1,\ldots,\pm(M-2) \tag{A-3b}$$

$$\underset{\sim}{\phi}_{2-M} + A\underset{\sim}{\phi}_{1-M} = \underset{\sim}{Y}_{1-M} \; . \tag{A-3c}$$

Then for $j = \ell-1, \ell, \ell+1$ where $\ell = -M+2, \ldots, M-2$, Eq. (A-3b) can be written

$$\begin{aligned}
\underset{\sim}{\phi}_{\ell+2} + A\underset{\sim}{\phi}_{\ell+1} + \underset{\sim}{\phi}_{\ell} &= 0 \\
\underset{\sim}{\phi}_{\ell+1} + A\underset{\sim}{\phi}_{\ell} + \underset{\sim}{\phi}_{\ell-1} &= 0 \\
\underset{\sim}{\phi}_{\ell} + A\underset{\sim}{\phi}_{\ell-1} + \phi_{\ell-2} &= 0 \; .
\end{aligned}$$

Then multiplying the middle equation by $-A$ and adding the three equations yields

$$\underset{\sim}{\phi}_{\ell+2} + (2I-A^2)\underset{\sim}{\phi}_{\ell} + \underset{\sim}{\phi}_{\ell-2} = 0$$

$$\text{for } \ell = -M+2, \ldots, M-2 \; .$$

Buneman, as described by Hockney (Ref. 7) proceeds by these steps and then reapplies the method. Thus with

$$A_1 = 2I-A^2 \; ,$$

$$\begin{aligned}
\underset{\sim}{\phi}_{\ell+4} + A_1\underset{\sim}{\phi}_{\ell+2} + \underset{\sim}{\phi}_{\ell} &= 0 \\
\underset{\sim}{\phi}_{\ell+2} + A_1\underset{\sim}{\phi}_{\ell} + \underset{\sim}{\phi}_{\ell-2} &= 0 \\
\underset{\sim}{\phi}_{\ell} + A_1\underset{\sim}{\phi}_{\ell-2} + \underset{\sim}{\phi}_{\ell-4} &= 0
\end{aligned}$$

where $\ell = -M+4, \ldots, M-4$

and again, multiplying the middle equation by $-A$, and adding yields

$$\underset{\sim}{\phi}_{\ell+4} + (2I-A_1^2)\underset{\sim}{\phi}_{\ell} + \underset{\sim}{\phi}_{\ell-4} = 0 \; .$$

Then repeating the process of cyclic reduction recursively, Buneman obtains for the i^{th} recursion

$$\underset{\sim}{\phi}_{j+2^i} + A_i \underset{\sim}{\phi}_j + \underset{\sim}{\phi}_{j-2^i} = 0$$

$$A_i = 2I - A_{i-1}^2 \qquad \text{(A-4)}$$

$$A_0 = A$$

Hence, when j = 0 and i = k there results

$$\underset{\sim}{\phi}_M + A_K \underset{\sim}{\phi}_0 + \underset{\sim}{\phi}_{-M} = 0$$

so that

$$\underset{\sim}{\phi}_0 = - A_K^{-1} (\underset{\sim}{\phi}_M + \underset{\sim}{\phi}_{-M}) \ .$$

$\underset{\sim}{\phi}_M$ and $\underset{\sim}{\phi}_{-M}$ are known values from Eq. (A-1); hence, $\underset{\sim}{\phi}_0$ may be found by inverting an (N-1) x (N-1) matrix. Once $\underset{\sim}{\phi}_0$ is known, the method may be repeated on the regions

$$R_U = \left\{ \begin{array}{c} (x,y) \quad - x^* \leq x \leq x^* \\ \\ 0 \leq y \leq 1 \end{array} \right\} \text{ and } R_L = \left\{ \begin{array}{c} (x,y) \quad - x^* \leq x \leq x^* \\ \\ -1 \leq y \leq 0 \end{array} \right\}$$

solving for $\underset{\sim}{\phi}_{\frac{M}{2}}$ and $\underset{\sim}{\phi}_{-\frac{M}{2}}$.

These steps are repeated until all the vectors $\underset{\sim}{\phi}_\ell$ $\ell = 0, \pm 1, \ldots, \pm(M-1)$ are found. Each step requires finding the solution to N-1 linear equations.

APPENDIX B
MODIFICATION OF BLOCK CYCLIC REDUCTION

The set of finite difference equations (Eqs. (8), (11), and (12)) was developed in Section 2.2 and is given by

$$\underset{\sim}{\phi}_{j+1} + A\underset{\sim}{\phi}_j + \underset{\sim}{\phi}_{j-1} = 0$$

$$j = 0, \pm 1, \ldots , (M-1) \qquad \text{(B-1)}$$

$$T\underset{\sim}{\phi}_{M-1} = E\underset{\sim}{\phi}_M + \underset{\sim}{f}^+ \qquad \text{(B-2a)}$$

$$T\underset{\sim}{\phi}_{1-M} = E\underset{\sim}{\phi}_{-M} + \underset{\sim}{f}^- \ . \qquad \text{(B-2b)}$$

It will be shown that the vectors $\underset{\sim}{\phi}_M$ and $\underset{\sim}{\phi}_{-M}$ can be found as the solution to a system of 2(N-1) linear equations.

At this point, a change of notation will be made for convenience.

Let

$$\underset{\sim}{V}_{\ell+M} = \underset{\sim}{\phi}_\ell \qquad \ell = -M, \ldots , M \ . \qquad \text{(B-3)}$$

Applying Eq. (B-3) to Eq. (A-4) results in

$$\underset{\sim}{V}_{j+2^i+M} + A_i \underset{\sim}{V}_{j+M} + \underset{\sim}{V}_{j-2^i+M} = 0 \qquad \text{(B-4)}$$

Theorem I

$$\underset{\sim}{V}_{\ell+1} = F_n \underset{\sim}{V}_\ell + G_n \underset{\sim}{V}_{\ell+2^n} \qquad \text{(B-5)}$$

where

$$n = 1, 2, \ldots , K+1$$

and ℓ is such that

$$\ell \geq 0 \text{ and } 2^n + \ell \leq 2M ,$$

where

$$F_n = F_{n-1} - G_{n-1} A_{n-1}^{-1} \quad n = 2, \ldots , K+1 \tag{B-6}$$

$$G_n = - G_{n-1} A_{n-1}^{-1} \quad n = 2, \ldots , K+1 \tag{B-7}$$

and where

$$F_1 = - A_0^{-1} , \; G_1 = - A_0^{-1} .$$

Proof

In Eq. (B-4) let $j = 1 - m + \ell$ and $i = 0$.

Then

$$\underset{\sim}{V}_{2+\ell} + A_0 \underset{\sim}{V}_{1+\ell} + \underset{\sim}{V}_\ell = 0$$

hence

$$\begin{aligned} \underset{\sim}{V}_{\ell+1} &= - A_0^{-1} (\underset{\sim}{V}_\ell + \underset{\sim}{V}_{\ell+2}) \\ &= F_1 \underset{\sim}{V}_\ell + G_1 \underset{\sim}{V}_{\ell+2} \end{aligned}$$

so the theorem holds for $n = 1$. These steps would complete the proof for $K = 0$ so now assume $K > 0$ and suppose Eq. (B-5) holds for $n = L$, $L = 1, \ldots , K$.

Then

$$\underset{\sim}{V}_{\ell+1} = F_L \underset{\sim}{V}_\ell + G_L \underset{\sim}{V}_{\ell+2^L} \tag{B-8}$$

where ℓ is such that

$$\ell \geq 0 \text{ and } 2^L + \ell \leq M .$$

In Eq. (B-4) then, let $j = 2^L - 2^K + \ell$ and let $i = L$.

Then

$$\underset{\sim}{V}_{2^{L+1}+\ell} + A_L \underset{\sim}{V}_{2^L+\ell} + \underset{\sim}{V}_\ell = 0$$

or

$$\underset{\sim}{V}_{2^{L}+\ell} = - A_L^{-1}\left(\underset{\sim}{V}_\ell + \underset{\sim}{V}_{2^{L+1}+\ell}\right) . \tag{B-9}$$

Substituting Eq. (B-9) into the inductive hypothesis Eq. (B-8) gives

$$\begin{aligned}\underset{\sim}{V}_{\ell+1} &= F_L \underset{\sim}{V}_\ell + G_L\left[- A_L^{-1}\left(\underset{\sim}{V}_\ell + \underset{\sim}{V}_{2^{L+1}+\ell}\right)\right] \\ &= \left(F_L - G_L A_L^{-1}\right)\underset{\sim}{V}_\ell - G_L A_L^{-1} \underset{\sim}{V}_{2^{L+1}+\ell} \\ &= F_{L+1} \underset{\sim}{V}_\ell + G_{L+1} \underset{\sim}{V}_{2^{L+1}+\ell}\end{aligned}$$

and hence the proof is complete.

Then with $n = K+1$ and $\ell = 0$. Eq. (B-5) becomes

$$\begin{aligned}\underset{\sim}{V}_1 &= F_{K+1} \underset{\sim}{V}_0 + G_{K+1} \underset{\sim}{V}_{2^{K+1}} \\ &= F_{K+1} \underset{\sim}{V}_0 + G_{K+1} \underset{\sim}{V}_{2M} .\end{aligned} \tag{B-10}$$

In a similar manner, it may be shown that

$$\underset{\sim}{V}_{2M-1} = F_{K+1} \underset{\sim}{V}_{2M} + G_{K+1} \underset{\sim}{V}_0 . \tag{B-11}$$

Applying Eq. (B-3) to Eqs. (B-10) and (B-11) gives

$$\underset{\sim}{\phi}_{1-M} = F_{K+1} \underset{\sim}{\phi}_{-M} + G_{K+1} \underset{\sim}{\phi}_M \tag{B-12}$$

$$\underset{\sim}{\phi}_{M-1} = F_{K+1} \underset{\sim}{\phi}_M + G_{K+1} \underset{\sim}{\phi}_{-M} . \tag{B-13}$$

Substituting Eqs. (B-10) and (B-11) into Eqs. (B-2a) and (B-2b) respectively, yields

$$T\left[F_{K+1} \underset{\sim}{\phi}_M + G_{K+1} \underset{\sim}{\phi}_{-M}\right] = E\underset{\sim}{\phi}_M + \underset{\sim}{f}^{+} \tag{B-14}$$

$$T\left[F_{K+1} \underset{\sim}{\phi}_{-M} + G_{K+1} \underset{\sim}{\phi}_M\right] = E\underset{\sim}{\phi}_{-M} + \underset{\sim}{f}^{-} \tag{B-15}$$

which may be written in Block Matrix form as

$$\begin{bmatrix} TF_{K+1} - E & G_{K+1} \\ G_{K+1} & TF_{K+1} - E \end{bmatrix} \begin{bmatrix} \underset{\sim}{\phi}_M \\ \underset{\sim}{\phi}_{-M} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{f}^+ \\ \underset{\sim}{f}^- \end{bmatrix} . \qquad \text{(B-16)}$$

It will be noted that Eq. (B-16) is a linear system of 2(N-1) equations which can be solved for $\underset{\sim}{\phi}_M$ and $\underset{\sim}{\phi}_{-M}$. Once these vectors are known, the problem becomes one of the Dirchlet type which can be solved by the methods of Appendix A.

In Theorem I, let n = K and ℓ = M; then Eq. (B-5) becomes

$$\underset{\sim}{V}_{M+1} = F_K \underset{\sim}{V}_M + G_K \underset{\sim}{V}_{2M} . \qquad \text{(B-17)}$$

Also, writing Eq. (B-4) with j = 0, i = K and again noting that $2^K = M$ results in

$$\underset{\sim}{V}_{2M} + A_K \underset{\sim}{V}_M + \underset{\sim}{V}_0 = 0$$

or

$$\underset{\sim}{V}_M = - A_K^{-1} (\underset{\sim}{V}_0 + \underset{\sim}{V}_{2M}) . \qquad \text{(B-18)}$$

Substituting Eq. (B-18) into Eq. (B-17) gives

$$\underset{\sim}{V}_{M+1} = F_K \left[- A_K^{-1} \left(\underset{\sim}{V}_0 + \underset{\sim}{V}_{2M}\right)\right] + G_K \underset{\sim}{V}_{2M}$$

$$= \left(G_K - F_K A_K^{-1}\right) \underset{\sim}{V}_{2M} - F_K A_K^{-1} V_0 .$$

Letting

$$S = G_K - F_K A_K^{-1} \text{ and } W = - F_K A_K^{-1}$$

gives

$$\underset{\sim}{V}_{M+1} = S \underset{\sim}{V}_{2M} + W \underset{\sim}{V}_0$$

and then by use of Eq. (B-3)

$$\underset{\sim}{\phi}_1 = S \underset{\sim}{\phi}_M + W \underset{\sim}{\phi}_{-M} .$$

In a similar manner it is found that

$$\underset{\sim}{\phi}_{-1} = W\underset{\sim}{\phi}_{M} + S\underset{\sim}{\phi}_{-M} \; .$$

Then

$$\underset{\sim}{\phi}_{1} - \underset{\sim}{\phi}_{-1} = (S-W)\;(\underset{\sim}{\phi}_{M} - \underset{\sim}{\phi}_{-M}) \; . \qquad \text{(B-19)}$$

Subtracting Eq. (B-15) from Eq. (B-14) results in

$$(TF_{K+1} - TG_{K+1} - E)\;(\underset{\sim}{\phi}_{M} - \underset{\sim}{\phi}_{-M}) = (\underset{\sim}{f}^{+} - \underset{\sim}{f}^{-})$$

so that

$$\underset{\sim}{\phi}_{1} - \underset{\sim}{\phi}_{-1} = (S-W)\left(TF_{K+1} - TG_{K+1} - E\right)^{-1}(\underset{\sim}{f}^{+} - \underset{\sim}{f}^{-}) \qquad \text{(B-20)}$$

hence, $\underset{\sim}{\phi}_{1} - \underset{\sim}{\phi}_{-1}$ can be computed by solving an N-1 system of equations and the interference factor in Eq. (13) is obtained.

APPENDIX C
METHOD OF EVALUATION

The evaluation of the lift interference by use of Eq. (B-20) is greatly hindered by the number of operations required to evaluate the recursion matrices F_{K+1} and G_{K+1}. However, these computations may be greatly simplified by a simple application of induction and it is shown by use of Eqs. (B-6) and (B-7) that

$$G_K = (-1)^K (A_{K-1} \cdots . A_1 A_0)^{-1} \qquad \text{(C-1)}$$

and

$$F_K = \sum_{\ell=1}^{K} G_\ell$$

so that

$$F_{K+1} - G_{K+1} = \sum_{\ell=1}^{K+1} G_\ell - G_{K+1} = F_K \ .$$

Since $S - W = G_K$, Eq. (B-20) becomes

$$\underset{\sim}{\phi}_1 - \underset{\sim}{\phi}_{-1} = G_K (TF_K - E)^{-1} (\underset{\sim}{f}^+ - \underset{\sim}{f}^-) \ .$$

Now consider the matrix A given by Eq. (9).

Define the matrix

$$D = \left[0,\ d_i,\ 0\right]_{N-1 \times N-1}$$

where $d_1 = 1$ and $d_{j+1} = a_j\ d_j / C_{j+1}$, $j = 1, \ldots, N-2$. (C-2)

Define $\hat{A} = D^{\frac{1}{2}}\ A\ D^{-\frac{1}{2}} = \left[\hat{c}_i,\ \hat{b}_i,\ \hat{a}_i\right]_{N-1 \times N-1}$ (C-3)

where $\hat{b}_i = b_i \qquad i = 1, \ldots, N-1$

and $\hat{c}_i = \hat{a}_{i-1} = \sqrt{c_i\ a_{i-1}} \qquad i = 2, \ldots, N-1.$

Then $\hat{A}$ is a real symmetric matrix for which many well known computer programs can be used to compute the eigenvalues and eigenvectors λ_j and $\underset{\sim}{x}_j \cdot j = 1, \ldots, N-1$.

From Eq. (C-3), $\hat{A}$ and A are seen to be similar matrices hence λ_j and $D^{-1/2}\underset{\sim}{x}_j$ form an eigenvalue, eigenvector pair for A.

Let X be the unitary matrix whose columns are the set of orthonormal eigenvectors of $\hat{A}$.

Then $X^{-1} \hat{A} X = \Lambda$

where $\Lambda = \left[0, \lambda_i, 0\right]_{N-1 \times N-1}$.

Hence $Q^{-1} A Q = \Lambda$ where $Q = D^{-1/2} X$.

Now suppose there exists a matrix Q_k which diagonalizes A_k so that $Q_k^{-1} A_k Q_k = \Lambda_k$.

Then since $A_{k+1} = 2I - A_k^2$,

$$
\begin{aligned}
Q_k^{-1} A_{k+1} Q_k &= Q_k^{-1} 2I Q_k - Q_k^{-1} A_k^2 Q_k \\
&= 2I - Q_k^{-1} A_k Q_k Q_k^{-1} A_k Q_k \\
&= 2I - \Lambda_k^2 .
\end{aligned}
$$

Hence, Q_k diagonalizes both Q_k and Q_{k+1}. Then dropping the subscript on Q gives

$$
Q^{-1} A_k Q = \Lambda_k
$$

so that from Eq. (C-1)

$$
\begin{aligned}
G_k &= (-1)^k (A_{k-1} \ldots . A_1 A_o)^{-1} \\
&= (-1)^k (Q\Lambda_{k-1} Q^{-1} Q\Lambda_{k-2} Q^{-1} \ldots Q\Lambda_o Q^{-1}) \\
&= (-1)^k (Q\Lambda_{k-1} \ldots . \Lambda_1 \Lambda_o Q^{-1})
\end{aligned}
$$

But $Q^{-1} = (D^{-1/2}X)^{-1} = X^{-1}\,D^{1/2}$

and since X is a unitary matrix,

$$Q^{-1} = X^T\,D^{\frac{1}{2}}\ . \qquad \text{So finally,}$$

$$G_k = (-1)^k\,(D^{-\frac{1}{2}}\,X\,\lambda_{k-1}\,\ldots.\lambda_1\,\lambda_o\,X^T\,D^{\frac{1}{2}})\ .$$

Hence the work of computing G_k is simplified since most matrices involved are diagonal matrices.

It will be noted that G_k and F_k are functions only of the mesh and hence the interference may be computed via Eq. (C-2) for many different porosity distributions T without having to recompute these matrices.

APPENDIX D
COMPUTER PROGRAM

Program Description

MAIN

The main program is for control purposes. SETTUP should be called immediately. TFIX is called whenever a new porosity distribution is desired. EVALU8 is called to compute the lift interference. In the sample listing, the interference is computed for the four porosity distributions given in Table 1.

B

Function B is a user-supplied routine and is used to compute

$$B_i^{\pm} = -\frac{R(x_i)}{\beta}\frac{\partial\phi m(x_i, \pm 1)}{\partial x} \pm \frac{\partial\phi m(x_i, \pm 1)}{\partial y} .$$

This equation is similar to the one following Eq. (9). It need be rewritten only when the model velocity potential is changed.

CHOLES

Subroutine CHOLES solves the linear system given by Eq. (C-2) by the method of matrix factorization.

DFFIX

Subroutine DFFIX computes the vector $\underset{\sim}{f}^{+} - \underset{\sim}{f}^{-}$.

EVALU8

This subroutine evaluates the finite difference coefficients, constructs the matrix $(TF_k - E)\, G_k^{-1}$, and calls subroutine CHOLES to solve Eq. (C-2).

FFIX

This subroutine constructs the matrices F_k and G_k by the methods described in Appendix C.

HDIAG

Subroutine HDIAG computes the eigenvectors and eigenvalues of the matrix $\hat{A}$ defined by Eq. (C-3).

MESH

This user-supplied subroutine is used to fill the X and DX arrays. Note that a value is assigned to X(o), namely X(o) = -X*.

MULT and MULT2

These routines perform the FORTRAN matrix replacements B = AB and A = AB, respectively.

SETTUP

This is a user-supplied routine used to initialize all program constants.

TFIX

TFIX loads the vector $T(x_i)$ by calling TFUNC.

TFUNC

This is a user-supplied routine used to evaluate the function.

$$\text{TFUNC}\ (x) = R(x)/\beta$$

It should be noted here that in the program the array T contains values of R/β and not β/R as given following Eq. (9).

MAIN

```
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON T(49),VEC(49),DUM(49,50)
      COMMON/XX/XD,X(50)
      COMMON/I1/N,M,N1,M1,K
      COMMON/CONTRL/IC
      CALL SETTUP
      DO 50 ICC=1,4
      IC=5-ICC
      CALL TFIX
      CALL EVALU8
      DO 39 I=1,N1
39    WRITE(6,10)I,X(I),T(I),DUM(I,N)
10    FORMAT(5X,'I=',I3,5X,'X=',F10.3,5X,'BETA/RO=',D16.8,5X,
     *'DELTA=',D16.8)
50    CONTINUE
      STOP
      END
```

B

```
      DOUBLE PRECISION FUNCTION B(J)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/R1/DXO,DX(49),DUM(5),XINF,PI2
      COMMON/XX/XD,EX(50)
      COMMON T(49)
      COMMON/BROWN/GAM
      I=IABS(J)
      X=EX(I)
      Y=I/J
      B=(-Y*T(I)+Y*X)/(X*X+1.D0)*GAM/PI2
      RETURN
      END
```

CHOLES

```
      SUBROUTINE CHOLES(A,N,NV,ID1,ID2,MATSYM)
      REAL*8 A(ID1,ID2),SUM,TEMP
      M=N+NV
      NARD=N+1
      IF(A(1,1).NE.0.0) GO TO 47
      DO 37 J=2,N
      IF(A(J,1).EQ.0.0) GO TO 37
      IFLIP=J
      GO TO 27
   37 CONTINUE
      GO TO 54321
   27 DO 57 K=1,M
      TEMP=A(IFLIP,K)
      A(IFLIP,K)=A(1,K)
      A(1,K)=TEMP
   57 CONTINUE
   47 DO 2 J=2,M
      A(1,J)=A(1,J)/A(1,1)
    2 CONTINUE
      DO 6 I=2,N
      DO 7 J=2,M
      IF(MATSYM.EQ.0)GO TO 49
      IF(I-J)69,68,67
   49 IF(J.GT.1)GO TO 69
   68 K=J-1
      SUM=0.0
      DO 3 IR=1,K
      SUM=SUM+A(I,IR)*A(IR,J)
    3 CONTINUE
      A(I,J)=A(I,J)-SUM
      GO TO 7
   69 K=I-1
      SUM=0.0
      DO 4 IR=1,K
      SUM=SUM+A(I,IR)*A(IR,J)
    4 CONTINUE
      IF(A(I,I).EQ.0.0) GO TO 54321
      A(I,J)=(A(I,J)-SUM)/A(I,I)
      GO TO 7
   67 A(I,J)=A(J,I)*A(J,J)
    7 CONTINUE
    6 CONTINUE
      DO 52 NPROB=NARD,M
      DO 52 K=2,N
      I=N+1-K
      SUM=0.0
      LL=I+1
      DO 51 IR=LL,N
      SUM=SUM+A(I,IR)*A(IR,NPROB)
   51 CONTINUE
      A(I,NPROB)=A(I,NPROB)-SUM
   52 CONTINUE
      GO TO 12345
54321 N=-1
12345 RETURN
      END
```

DFFIX

```
      SUBROUTINE DFFIX
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON T(49),VEC(49)
      COMMON/I1/N,M,N1
      COMMON/R1/DXO,DX(49),DY
      DO 1 I=1,N1
    1 VEC(I)=-DY*(B(I)-B(-1))
      RETURN
      END
```

EVALU8

```
      SUBROUTINE EVALU8
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON T(49),VEC(49),DUM(49,50),FK(49,49),APROD(49,49)
      COMMON/R1/DXO,DX(49),DY,BETA,RO,W,CS,XINF,PI2,Y,DY2
      COMMON/I1/N,M,N1,M1,K,ID1,ID2,IZ,N2
      COMMON/ABC/A(49),B(49),C(49)
  100 CALL DFFIX
      DO 1 I=1,N1
      DO 1 J=1,N1
    1 DUM(I,J)=FK(I,J)
      Q1=DY2*(DX(1)-DX(IZ))/(DX(1)*DX(IZ))
      P1=DY2*DX(IZ)/(DX(1)*(DX(1)+DX(IZ)))
      DUM(1,1)=DUM(1,1)-.5D0*(T(1)*Q1-B(1))
      DUM(1,2)=DUM(1,2)-.5D0*(T(1)*P1-A(1))
      DO 65 I=2,N2
      QI=DY2*(DX(I)-DX(I-1))/(DX(I)*DX(I-1))
      PI=DY2*DX(I-1)/(DX(I)*(DX(I)+DX(I-1)))
      RI=-DY2*DX(I)/(DX(I-1)*(DX(I)+DX(I-1)))
      DUM(I,I-1)=DUM(I,I-1)-.5D0*(T(I)*RI-C(I))
      DUM(I,I)=DUM(I,I)-.5D0*(T(I)*QI-B(I))
   65 DUM(I,I+1)=DUM(I,I+1)-.5D0*(T(I)*PI-A(I))
      R1=-DY2*DX(N1)/(DX(N2)*(DX(N1)+DX(N2)))
      Q1=DY2*(DX(N1)-DX(N2))/(DX(N1)*DX(N2))
      DUM(N1,N2)=DUM(N1,N2)-.5D0*(T(N1)*R1-C(N1))
      DUM(N1,N1)=DUM(N1,N1)-.5D0*(T(N1)*Q1-B(N1))
      CALL MULT2(DUM,APROD)
      DO 20 I=1,N1
   20 DUM(I,N)=VEC(I)
      CALL CHOLES(DUM,N1,1,ID1,ID2,0)
      DO 4 I=1,N1
    4 DUM(I,N)=DUM(I,N)/(2.D0*DY)
      RETURN
      END
```

FFIX

```
      SUBROUTINE FFIX
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON T(49),VEC(49),TMP(49,50),FK(49,49),APROD(49,49)
      COMMON/R1/DXO,DX(49),DY,DUM(5),PI2
      DIMENSION D(49),DG(49),DF(49)
      COMMON/I1/N,M,N1,M1,K
      COMMON/ABC/A(49),B(49),C(49)
      DY2=2.D0*DY*DY
      DO 20 I=1,N1
      A(I)=DY2/(DX(I)*(DX(I)+DX(I-1)))
      B(I)=-2.D0-DY2/(DX(I)*DX(I-1))
   20 C(I)=DY2/(DX(I-1)*(DX(I)+DX(I-1)))
      D(1)=1.D0
      N2=N1-1
      DO 30 I=1,N2
   30 D(I+1)=A(I)*D(I)/C(I+1)
      DO 60 I=1,N1
      DO 60 J=1,N1
   60 FK(I,J)=0.D0
      FK(1,1)=B(1)
      DO 40 I=2,N1
      FK(I,I)=B(I)
      FK(I,I-1)=DSQRT(C(I)*A(I-1))
   40 FK(I-1,I)=FK(I,I-1)
      IEGN=0
      CALL HDIAG(FK,N1,IEGN,TMP,NRN)
      DO 50 I=1,N1
      VEC(I)=1.D0/DSQRT(D(I))
   50 D(I)=FK(I,I)
      DO 1 I=1,N1
      DF(I)=0.D0
    1 DG(I)=1.D0
      DO 2 IK=1,K
      DO 2 I=1,N1
      DG(I)=D(I)*DG(I)
      DF(I)=(-1)**IK/DG(I)+DF(I)
    2 D(I)=2.D0-D(I)**2
      EE=(-1)**K
      DO 10 I=1,N1
   10 DG(I)=EE*DG(I)
      DO 501 J=1,N1
      DO 501 I=1,N1
      APROD(I,J)=TMP(I,J)*DG(J)
  501 FK(I,J)=TMP(I,J)*DF(J)
      DO 502 I=1,N1
      DO 502 J=1,N1
      APROD(I,J)=VEC(I)*APROD(I,J)
```

FFIX

```
502 FK(I,J)=VEC(I)*FK(I,J)
    DO 503 I=2,N1
    IM1=I-1
    DO 503 J=1,IM1
    TEMP=TMP(I,J)
    TMP(I,J)=TMP(J,I)
503 TMP(J,I)=TEMP
    CALL MULT2(APROD,TMP)
    CALL MULT2(FK,TMP)
    DO 504 J=1,N1
    TOM=1.DO/VEC(J)
    DO 504 I=1,N1
    APROD(I,J)=APROD(I,J)*TOM
504 FK(I,J)=FK(I,J)*TOM
    RETURN
    END
```

HDIAG

```
      SUBROUTINE  HDIAG (H,N,IEGEN,U,NR)
C     SUBROUTINE HDIAG.
C
C     PROGRAMED BY F. J. CARBATO AND M.MERWIN OF THE MIT
C     COMPUTATION CENTER.
C
C          THIS SUBROUTINE COMPUTES THE EIGENVALUES AND EIGENVECTORS
C     OF A REAL SYMMETRIC MATRIX, H, OF ORDER N ( WHERE N MUST BE LESS
C     THAN 51), AND PLACES THE EIGENVALUES IN THE DIAGONAL ELEMENTS OF
C     THE MATRIX H, AND PLACES THE EIGENVECTORS (NORMALIZED ) IN THE
C     COLUMNS OF THE MATRIX U. IEGEN IS SET AS 1 IF ONLY EIGENVALUES
C     ARE DESIRED,AND IS SET TO 0 WHEN VECTORS ARE REQUIRED. NR CON-
C     TAINS THE NUMBER OF ROTATIONS DONE.
C
C     H, N, IENGEN, U, AND NR OF THE ARGUMENT LIST ARE DUMMY VARIABLES
C     AND MAY BE NAMED DIFFERENTLY IN THE CALLING OF THE SUBROUTINE.
C
C     SUBROUTINE PLACES COMPUTER IN THE FLOATING TRAP MODE
C     THE SUBROUTINE OPERATES ONLY ON THE ELEMENTS OF H THAT ARE TO THE
C            RIGHT OF THE MAIN DIAGONAL. THUS, ONLY A TRIANGULAR
C            SECTION NEED BE STORED IN THE ARRAY H.
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION H(49,49),U(49,49),X(49),IQ(49)
    2 FORMAT(14H MAX OFF DIAG=,F14.7,3HNR=,I3)
 2001 FORMAT(1X,8E15.8)
 2002 FORMAT(18H ORTHOGONAL MATRIX)
 2003 FORMAT(15H ROTATED MATRIX)
      SIGN(X1,X2)=DSIGN(X1,X2)
      SQRT(X)=DSQRT(X)
      ABS(X)=DABS(X)
      IF(IEGEN.NE.0) GO TO 15
   10 DO 14 I=1,N
      DO 14 J=1,N
      IF(I-J.NE.0) GO TO 12
   11 U(I,J)=1.0
      GO TO 14
   12 U(I,J)=0.0
   14 CONTINUE
   15 NR = 0
      IF(N-1.LE.0)  GO TO 1000
C     SCAN FOR LARGEST OFF-DIAGONAL ELEMENT IN EACH ROW
C     X(I) CONTAINS LARGEST ELEMENT IN ITH ROW
C     IQ(I) HOLDS SECOND SUBSCRIPT DEFINING POSITION OF ELEMENT
   17 NMI1=N-1
      DO 30 I=1,NMI1
      X(I) = 0.0
      IPL1=I+1
      DO 30 J=IPL1,N
      IF(X(I)-ABS(H(I,J)).GT. 0.0) GO TO 30
   20 X(I)=ABS(H(I,J))
      IQ(I)=J
   30 CONTINUE
C     SET INDICATOR FOR SHUT-OFF.RAP=2**-27,NR=NO. OF ROTATIONS
      RAP=7.450580596D-9
      HDTEST=1.0D38
C     FIND MAXIMUM OF X(I) S FOR PIVOT ELEMENT AND
```

HDIAG

```
C       TEST FOR END OF PROBLEM
   40 DO 70 I=1,NMI1
        IF(I-1.LE.0) GO TO 60
        IF(XMAX-X(I).GE.0.0) GO TO 70
   60 XMAX=X(I)
        IPIV=I
        JPIV=IQ(I)
   70 CONTINUE
C       IS MAX. X(I) EQUAL TO ZERO, IF LESS THAN HDTEST, REVISE HDTEST
        IF(XMAX.LE.0.0) GO TO 1002
   80 IF(HDTEST.LE.0.0) GO TO 90
   85 IF(XMAX-HDTEST.GT.0.0) GO TO 148
   90 HDIMIN = ABS( H(1,1) )
        DO 110 I=2,N
        IF(HDIMIN- ABS( H(I,I) ).LE. 0.0) GO TO 110
  100 HDIMIN=ABS( H(I,I) )
  110 CONTINUE
        HDTEST = HDIMIN*RAP
C       RETURN IF MAX.H(I,J)LESS THAN(2**-27)ABS(H(K,K)-MIN)
        IF(HDTEST-XMAX.GE.0.0)GO TO 1002
  148 NR = NR+1
C COMPUTE TANGENT, SINE AND COSINE,H(I,I),H(J,J)
  150 TANG=SIGN(2.0,(H(IPIV,IPIV)-H(JPIV,JPIV)))*H(IPIV,JPIV)/(ABS(H(I
     1PIV,IPIV)-H(JPIV,JPIV))+SQRT((H(IPIV,IPIV)-H(JPIV,JPIV))**2+4.0*H
     2(IPIV,JPIV)**2))
        COSINE=1.0/SQRT(1.0+TANG**2)
        SINE=TANG*COSINE
        HII=H(IPIV,IPIV)
        H(IPIV,IPIV)=COSINE**2*(HII+TANG*(2.*H(IPIV,JPIV)+TANG*H(JPIV,
     1JPIV)))
        H(JPIV,JPIV)=COSINE**2*(H(JPIV,JPIV)-TANG*(2.*H(IPIV,JPIV)-TANG*H
     1II))
        H(IPIV,JPIV)=0.0
C       PSEUDO RANK THE EIGENVALUES
C       ADJUST SINE AND COS FOR COMPUTATION OF H(IK) AND U(IK)
        IF(H(IPIV,IPIV)-H(JPIV,JPIV).GE.0.0) GO TO 153
  152 HTEMP = H(IPIV,IPIV)
        H(IPIV,IPIV) = H(JPIV,JPIV)
        H(JPIV,JPIV) = HTEMP
C       RECOMPUTE SINE AND COS
        HTEMP = SIGN(1.0, -SINE) * COSINE
        COSINE = ABS (SINE)
        SINE = HTEMP
  153 CONTINUE
C       INSPECT THE IQS BETWEEN I+1 AND N-1 TO DETERMINE
C       WHETHER A NEW MAXIMUM VALUE SHOULD BE COMPUTED SINCE
C       THE PRESENT MAXIMUM IS IN THE I OR J ROW.
        DO 350 I = 1,NMI1
        IF(I-IPIV.EQ.0) GO TO 350
        IF(I-IPIV.LT. 0 ) GO TO 210
  200 IF(I-JPIV.EQ. 0 ) GO TO 350
  210 IF(IQ(I)-IPIV.EQ. 0) GO TO 240
  230 IF(IQ(I)-JPIV.NE. 0 ) GO TO 350
  240 K = IQ(I)
  250 HTEMP = H(I,K)
        H(I,K) = 0.0
```

HDIAG

```
      IPL1 = I+1
      X(I) = 0.0
C     SEARCH IN DEPLETED ROW FOR NEW MAXIMUM
      DO 320 J = IPL1,N
      IF( X(I)-ABS( H(I,J) ).GT. 0.0) GO TO 320
  300 X(I) = ABS(H(I,J))
      IQ(I) = J
  320 CONTINUE
      H(I,K) = HTEMP
  350 CONTINUE
      X(IPIV) = 0.0
      X(JPIV) = 0.0
C     CHANGE THE OTHER ELEMENTS OF H
      DO 530 I = 1,N
      IF(I-IPIV.EQ. 0 ) GO TO 530
      IF(I-IPIV.GT. 0 ) GO TO 420
  370 HTEMP = H(I,IPIV)
      H(I,IPIV) = COSINE*HTEMP + SINE*H(I,JPIV)
      IF( X(I) - ABS( H(I,IPIV) ).GE. 0.0 ) GO TO 390
  380 X(I) = ABS( H(I,IPIV) )
      IQ(I) = IPIV
  390 H(I,JPIV) = -SINE*HTEMP + COSINE*H(I,JPIV)
      IF( X(I) - ABS( H(I,JPIV) ).GE. 0.0 ) GO TO 530
  400 X(I) = ABS( H(I,JPIV) )
      IQ(I) = JPIV
      GO TO 530
  420 IF( I-JPIV.EQ. 0 ) GO TO 530
      IF(I-JPIV.GT. 0 ) GO TU 480
  430 HTEMP = H(IPIV,I)
      H(IPIV,I) = COSINE*HTEMP + SINE*H(I,JPIV)
      IF( X(IPIV) - ABS( H(IPIV,I) ).GE. 0.0 ) GO TO 450
  440 X(IPIV) = ABS( H(IPIV,I) )
      IQ(IPIV) = I
  450 H(I,JPIV) = -SINE*HTEMP + COSINE*H(I,JPIV)
      IF( X(I) - ABS( H(I,JPIV) ).GE. 0.0 ) GO TO 530
      IF( X(I) - ABS( H(I,JPIV) ).LT. 0.0 ) GO TO 400
  480 HTEMP = H(IPIV,I)
      H(IPIV,I) = COSINE*HTEMP + SINE*H(JPIV,I)
      IF( X(IPIV) - ABS( H(IPIV,I) ).GE. 0.0 ) GO TO 500
  490 X(IPIV) = ABS( H(IPIV,I) )
      IQ(IPIV) = I
  500 H(JPIV,I) = -SINE*HTEMP + COSINE*H(JPIV,I)
      IF( X(JPIV) - ABS( H(JPIV,I) ).GE.0.0) GO TO 530
  510 X(JPIV) = ABS( H(JPIV,I) )
      IQ(JPIV) = I
  530 CONTINUE
C     TEST FOR COMPUTATION OF EIGENVECTORS
      IF(IEGEN.NE.0) GO TO 40
  540 DO 550 I = 1,N
      HTEMP = U(I,IPIV)
      U(I,IPIV) = COSINE*HTEMP + SINE*U(I,JPIV)
  550 U(I,JPIV) = -SINE*HTEMP+COSINE*U(I,JPIV)
      GO TO 40
 1002 CONTINUE
 1000 RETURN
      END
```

MESH

```
      SUBROUTINE MESH
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/R1/DXO,DX(49),DY,BETA,RO,C,CS,XINF
      COMMON/I1/N,M,N1
      COMMON/XX/XD,X(50)
      IZ=0
      X(IZ)=-XINF
      X(N)=XINF
      DO 1 I=1,5
    1 X(I)=-XINF+I*3.D0
      DO 2 I=6,44
    2 X(I)=-5.D0+(I-5)*.25D0
      DO 3 I=45,49
    3 X(I)=5.D0+(I-45)*3.D0
      DO 4 I=IZ,N1
    4 DX(I)=X(I+1)-X(I)
      RETURN
      END
```

MULT

```
      SUBROUTINE MULT(A,B)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/I1/N,M,N1
      DIMENSION A(49,1),B(49,1),TEMP(49)
      DO 1 J=1,N1
      DO 2 I=1,N1
      SUM=0.D0
      DO 3 K=1,N1
    3 SUM=SUM+A(I,K)*B(K,J)
    2 TEMP(I)=SUM
      DO 1 I=1,N1
    1 B(I,J)=TEMP(I)
      RETURN
      END
```

MULT2

```
      SUBROUTINE MULT2(A,B)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/I1/N,M,N1
      DIMENSION A(49,1),B(49,1),TEMP(49)
      DO 1 I=1,N1
      DO 2 J=1,N1
      SUM=0.D0
      DO 3 K=1,N1
    3 SUM=SUM+A(I,K)*B(K,J)
    2 TEMP(J)=SUM
      DO 1 J=1,N1
    1 A(I,J)=TEMP(J)
      RETURN
      END
```

SETTUP

```
      SUBROUTINE SETTUP
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/R1/DXO,DX(49),DY,BETA,RO,C,CS,XINF,PI2,Y,DY2
      COMMON/I1/N,M,N1,M1,K,ID1,ID2,IZ,N2
      COMMON/BROWN/GAM
C
C     THIS SUBROUTINE IS FOR INITIALIZING PROGRAM CONSTANTS
C
      Y=1.D0
      XINF=20.D0
      N=50
      M=16
      K=3
      BETA=4.D0
      RO=1.D0
      GAM=1.D0
      ID1=49
      ID2=50
      IZ=0
      N1=N-1
      N2=N-2
      DY=2.D0*Y/M
      DY2=2.D0*DY
      PI=3.1415 9265 3589 7932 D0
      PI2=2.D0*PI
      CALL MESH
      CALL FFIX
      RETURN
      END
```

TFIX

```
      SUBROUTINE TFIX
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON T(49)
      COMMON/I1/N,M,N1
      COMMON/R1/DXO,DX(49),DY,BETA,RO,C,CS,XINF
      COMMON/XX/XD,X(50)
      DO 2 I=1,N1
    2 T(I)=TFUNC(I,X(I))
      RETURN
      END
```

TFUNC

```
      DOUBLE PRECISION FUNCTION TFUNC(I,X)
      IMPLICIT REAL*8(A-H,O-Z)
      COMMON/CONTRL/IC
      TFUNC=DMAX1(X,0.D0)
      GO TO(1,2,3,4),IC
    1 RETURN
    2 IF(I.EQ.26)TFUNC=0.D0
      IF(I.EQ.27)TFUNC=.25D0
      IF(I.EQ.28)TFUNC=.5D0
      RETURN
    3 IF(I.EQ.26.OR.I.EQ.27)TFUNC=0.D0
      IF(I.EQ.28)TFUNC=.25D0
      IF(I.EQ.29)TFUNC=.5D0
      IF(I.EQ.30)TFUNC=.85D0
      IF(I.EQ.31)TFUNC=1.2D0
      IF(I.EQ.32)TFUNC=1.55D0
      RETURN
    4 IF(I.EQ.26.OR.I.EQ.27.OR.I.EQ.28)TFUNC=0.D0
      IF(I.EQ.29)TFUNC=.2D0
      IF(I.EQ.30)TFUNC=.45D0
      IF(I.EQ.31)TFUNC=.7D0
      IF(I.EQ.32)TFUNC=.975D0
      IF(I.EQ.33)TFUNC=1.3D0
      IF(I.EQ.34)TFUNC=1.6D0
      IF(I.EQ.35)TFUNC=1.95D0
      IF(I.EQ.36)TFUNC=2.375D0
      IF(I.EQ.37)TFUNC=2.85D0
      IF(I.EQ.38)TFUNC=3.25D0
      RETURN
      END
```

NOMENCLATURE

C	Airfoil chord length
C_L	Lift coefficient
h	Semiheight of tunnel
R(x)	Porosity parameter
S	Airfoil surface area
U	Free-stream velocity
x, y	Normalized Cartesian coordinates
X, Y	Cartesian coordinates (Fig. 1)
T	$\beta/R(x)$
β	Compressibility parameter
γ Γ	Vortex strength
δ	Lift interference factor, Eq. (13)
δx, δy	Finite spacing in x and y direction
Φ	Perturbation velocity potential
ϕ	Interference velocity potential
ϕ_m	Model velocity potential